

MOVING PARSEVAL FRAMES FOR VECTOR BUNDLES

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ABSTRACT. Parseval frames can be thought of as redundant or linearly dependent coordinate systems for Hilbert spaces, and have important applications in such areas as signal processing, data compression, and sampling theory. We extend the notion of a Parseval frame for a fixed Hilbert space to that of a moving Parseval frame for a vector bundle over a manifold. Many vector bundles do not have a moving basis, but in contrast to this every vector bundle over a paracompact manifold has a moving Parseval frame. We prove that a sequence of sections of a vector bundle is a moving Parseval frame if and only if the sections are the orthogonal projection of a moving orthonormal basis for a larger vector bundle. In the case that our vector bundle is the tangent bundle of a Riemannian manifold, we prove that a sequence of vector fields is a Parseval frame for the tangent bundle of a Riemannian manifold if and only if the vector fields are the orthogonal projection of a moving orthonormal basis for the tangent bundle of a larger Riemannian manifold.

1. INTRODUCTION

Frames for Hilbert spaces are essentially redundant coordinate systems. That is, every vector can be represented as a series of scaled frame vectors, but the series is not unique. Though this redundancy is not necessary in a coordinate system, it can actually be very useful. In particular, frames have played important roles in modern signal processing after originally being applied in 1986 by Daubechies, Grossmann, and Meyer [DGM]. Besides being important for their real world applications, frames are also interesting for both their analytic and geometric properties [BF],[BCPS],[DFKLOW],[HL] as well as their connection to the famous Kadison-Singer problem [CCLV],[W].

A sequence of vectors (x_i) in a Hilbert space H is called a *frame* for H if there exists constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum |\langle x_i, x \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in H.$$

The constants A, B are called the *frame bounds*. The frame is called *tight* if $A = B$, and is called *Parseval* if $A = B = 1$. The name Parseval was chosen because $A = B = 1$ if and only if the frame satisfies Parseval's identity. That is, a sequence of vectors (x_i) in a Hilbert space H is a Parseval frame for H if and only if

$$\sum \langle x_i, x \rangle x_i = x \text{ for all } x \in H.$$

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This useful reconstruction formula follows from the dilation theorem of Han and Larson [HL]. They proved that if (x_i) is a Parseval frame for a Hilbert space H , then (x_i) is the orthogonal projection of an orthonormal basis for a larger Hilbert space which contains H as a subspace. It is easy to see that the orthogonal projection of an orthonormal basis is a Parseval frame, and thus the dilation theorem characterizes Parseval frames as orthogonal projections of orthonormal bases.

In differential topology and differential geometry, the word frame has a different meaning. A moving frame for the tangent bundle of a smooth manifold is essentially a basis for the tangent space at each point in the manifold which varies smoothly over the manifold. In other words, a moving frame for the tangent bundle of an n -dimensional smooth manifold is a set of n linearly independent vector fields. These two different definitions for the word "frame", naturally lead one to question how they are related. We will combine the concepts by studying Parseval frames which vary smoothly over a manifold, which we formally define below.

Definition 1. Let $\pi : E \rightarrow M$ be a rank n -vector bundle over a smooth manifold M with a given inner product $\langle \cdot, \cdot \rangle$. Let $k \geq n$, and $f_i : M \rightarrow E$ be a smooth section of π for all $1 \leq i \leq k$. We say that $(f_i)_{i=1}^k$ is a *moving Parseval frame* for π if $(f_i(x))_{i=1}^k$ is a Parseval frame for the fiber $\pi^{-1}(x)$ for all $x \in M$. That is, for all $x \in M$,

$$y = \sum_{i=1}^k \langle y, f_i(x) \rangle f_i(x) \quad \text{for all } y \in \pi^{-1}(x).$$

A Parseval frame for a fixed Hilbert space can be constructed by projecting an orthonormal basis, and thus the natural way to construct moving Parseval frames is to project moving orthonormal bases. For instance, the two-dimensional sphere S^2 does not have a nowhere-zero vector field, and hence cannot have a moving orthonormal basis for its tangent space. However, if we consider S^2 as the unit sphere in \mathbb{R}^3 and $(e_i)_{i=1}^3$ as the standard unit vector basis for \mathbb{R}^3 , then at each point $p \in S^2$ we may project $(e_i)_{i=1}^3$ onto the tangent space $T_p(S^2)$, giving us a moving Parseval frame of three vectors for TS^2 . As every vector bundle over a paracompact manifold is a subbundle of a trivial bundle, we may project the basis for the trivial bundle onto the sub-bundle and obtain that every vector bundle over a paracompact manifold has a moving Parseval frame. Thus in contrast to moving bases, we have that moving Parseval frames always exist. The natural general questions to consider are then: When do moving Parseval frames with particular structure exist? How do theorems about Parseval frames generalize to the vector bundle setting?, and How can we construct nice moving Parseval frames for vector bundles in the absence of moving bases? Our main results are the following theorems which extend the dilation theorem of Han and Larson to the context of vector bundles. The proofs will be given in Section 3.

Theorem 2. Let $\pi_1 : E_1 \rightarrow M$ be a rank n vector bundle over a paracompact manifold M with a moving Parseval frame $(f_i)_{i=1}^k$. There exists a rank $k - n$ vector bundle $\pi_2 : E_2 \rightarrow M$ with a moving Parseval frame $(g_i)_{i=1}^k$ so that $(f_i \oplus g_i)_{i=1}^k$ is a moving orthonormal basis for the vector bundle $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$.

If M is a Riemannian manifold with a moving Parseval frame for TM , we may apply Theorem 2 to obtain a vector bundle containing TM with a moving orthonormal basis which projects to the moving Parseval frame. However, if we start with

a moving Parseval frame for a tangent bundle, we want to end up with a moving orthonormal basis for a larger tangent bundle which projects to the moving Parseval frame. This way we would remain in the class of tangent bundles, instead of general vector bundles. The following theorem states that we can do this.

Theorem 3. *Let M^n be an n -dimensional Riemannian manifold and $(f_i)_{i=1}^k$ be a moving Parseval frame for TM for some $k \geq n$. There exists a k -dimensional Riemannian manifold N^k with a moving orthonormal basis $(e_i)_{i=1}^k$ for TN such that N^k contains M^n as a submanifold and $P_{T_x M} e_i(x) = f_i(x)$ for all $x \in M^n$ and $1 \leq i \leq k$, where $P_{T_x M}$ is orthogonal projection from $T_x N$ onto $T_x M$.*

Though the concept of a moving Parseval frame seems natural to consider, we are aware of only one paper on the subject. In 2009, P. Kuchment proved in his Institute of Physics select paper that particular vector bundles over the torus, which arise in mathematical physics, have natural moving Parseval frames but do not have moving bases [K]. The relationship between frames for Hilbert spaces and manifolds was also considered in a different context by Dykema and Strawn, who studied the manifold structure of collections of Parseval frames under certain equivalent classes [DySt].

We will use the term *inner product* on a vector bundle $\pi : E \rightarrow M$ to mean a positive definite symmetric bilinear form. All of our theorems will concern vector bundles with a given inner product. In the case that our vector bundle is the tangent bundle of a Riemannian manifold, we will take the inner product to be the Riemannian metric. For terminology and background on vector bundles and smooth manifolds see [L], for terminology and background on frames for Hilbert spaces see [C] and [HKLW].

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2. PRELIMINARIES AND EXAMPLES

Our goal is to study moving Parseval frames and extend theorems about fixed Parseval frames for Hilbert spaces to moving Parseval frames for vector bundles. To do this, we will first need to define some notation and recall some useful characterizations of Parseval frames for Hilbert spaces in terms of matrices. For $F = (f_i)_{i=1}^k \in \oplus_{i=1}^k \mathbb{R}^n$ and $(u_i)_{i=1}^n$ a fixed orthonormal basis for \mathbb{R}^n , we denote $[F]_{n \times k}$ to be the matrix whose column vectors with respect to the basis $(u_i)_{i=1}^n$ are given by $(f_i)_{i=1}^k$. For $F = (f_i)_{i=1}^k \in \oplus_{i=1}^k \mathbb{R}^n$, $G = (g_i)_{i=1}^k \in \oplus_{i=1}^k \mathbb{R}^m$, we define $F \oplus G = (f_i \oplus g_i)_{i=1}^k \in \oplus_{i=1}^k \mathbb{R}^{n+m}$. If $k > n$, $(u_i)_{i=1}^n$ is a fixed orthonormal basis for \mathbb{R}^n and $(u_i)_{i=n+1}^k$ is a fixed orthonormal basis for \mathbb{R}^{k-n} , then the matrix $[F \oplus G]_{(n+m) \times k}$ given with respect to $(u_i)_{i=1}^k$ will be formed by appending the column vectors $(g_i)_{i=1}^k$ to the column vectors $(f_i)_{i=1}^k$. In other words, $[F \oplus G]_{(n+m) \times k} = \begin{pmatrix} f_1 & \cdots & f_k \\ g_1 & \cdots & g_k \end{pmatrix}$. This matrix framework allows us to provide a simple proof of the Han-Larson dilation theorem for Parseval frames for \mathbb{R}^n . In a later section we will extend this proof to vector bundles.

Theorem 4. [HL] *If $k > n$, and $(f_i)_{i=1}^k$ is a Parseval frame for \mathbb{R}^n , then there exists a Parseval frame $(g_i)_{i=1}^k$ for \mathbb{R}^{k-n} such that $(f_i \oplus g_i)_{i=1}^k$ is an orthonormal basis for $\mathbb{R}^n \oplus \mathbb{R}^{k-n}$.*

Proof. We denote the unit vector basis for \mathbb{R}^n by $(u_i)_{i=1}^n$. Let $F = (f_i)_{i=1}^k$ and let $[F]_{n \times k}$ be the matrix whose column vectors with respect to $(u_i)_{i=1}^n$ are given by $(f_i)_{i=1}^k$. If $1 \leq p, q \leq n$, then the inner product of the p th row of T with the q th row of T is given by $\sum_{i=1}^k \langle f_i, u_p \rangle \langle f_i, u_q \rangle$. We now use the following equality.

$$\begin{aligned} 2 &= \sum_{i=1}^k \langle f_i, u_p + u_q \rangle^2 = \sum_{i=1}^k \langle f_i, u_p \rangle^2 + \sum_{i=1}^k \langle f_i, u_q \rangle^2 + 2 \sum_{i=1}^k \langle f_i, u_p \rangle \langle f_i, u_q \rangle \\ &= 2 + 2 \sum_{i=1}^k \langle f_i, u_p \rangle \langle f_i, u_q \rangle \end{aligned}$$

Thus we have that $\sum_{i=1}^k \langle f_i, u_p \rangle \langle f_i, u_q \rangle = 0$, and hence the rows of $[F]_{n \times k}$ are orthonormal. We can thus choose $G = (g_i)_{i=1}^k \subset \mathbb{R}^{k-n}$ such that the rows of $[F \oplus G]_{k \times k}$ are orthonormal. Thus the column vectors $(f_i \oplus g_i)_{i=1}^k$ of $[F \oplus G]_{k \times k}$ form an orthonormal basis for $\mathbb{R}^n \oplus \mathbb{R}^{k-n}$. We have that $(g_i)_{i=1}^k$ must be a Parseval frame for \mathbb{R}^{k-n} as it is the orthogonal projection of the orthonormal basis $(f_i \oplus g_i)_{i=1}^k$. \square

As shown in the proof of Theorem 4, a sequence of vectors $F = (f_i)_{i=1}^k \subset \mathbb{R}^n$ is a Parseval frame for \mathbb{R}^n if and only if the matrix $[F]_{n \times k}$ has orthonormal rows. The dilation theorem gives that Parseval frames are exactly orthogonal projections of orthonormal bases. It is then immediate that the orthogonal projection of a moving orthonormal basis is a moving Parseval frame.

Theorem 5. *Let $k \geq n$ and let $\pi : E \rightarrow M$ be a rank k vector bundle with an inner product $\langle \cdot, \cdot \rangle$ and moving orthonormal basis $(e_i)_{i=1}^k$. If $\pi|_{E_0} : E_0 \rightarrow M$ is a rank n sub-bundle, then $(P_{E_0} e_i)_{i=1}^k$ is a moving Parseval frame for $\pi|_{E_0} : E_0 \rightarrow M$, where $P_{E_0}(e_i(x))$ is the orthogonal projection of $e_i(x)$ onto the fiber $\pi|_{E_0}^{-1}(x)$ for all $x \in M$.*

Proof. As $\pi|_{E_0} : E_0 \rightarrow M$ is a subbundle of $\pi : E \rightarrow M$, we have that $P_{E_0} : E \rightarrow E_0$ is continuous. Furthermore, for all $1 \leq i \leq k$, we have that $\pi|_{E_0}(P_{E_0}(e_i(x))) = x$ for all $x \in M$. Thus $P_{E_0} e_i$ is a section of $\pi|_{E_0} : E_0 \rightarrow M$ for all $1 \leq i \leq k$. $(P_{E_0} e_i(x))_{i=1}^k$ is a Parseval frame for $\pi|_{E_0}^{-1}(x)$ for all $x \in M$, as it is the orthogonal projection of an orthonormal basis. Thus $(P_{E_0} e_i(x))_{i=1}^k$ is a moving Parseval frame for $\pi|_{E_0} : E_0 \rightarrow M$. \square

By applying Theorem 5 to the tangent bundle of a smooth manifold, we obtain the following corollary for Riemannian manifolds.

Corollary 6. *Let $k \geq n$ and let N be a k -dimensional Riemannian manifold with a moving orthonormal basis $(e_i)_{i=1}^k$ for its tangent bundle TN . If $M \subset N$ is a smooth sub-manifold, then $(P_{TM} e_i)_{i=1}^k$ is a moving Parseval frame for TM , where $P_{TM}(e_i(x))$ is the orthogonal projection of $e_i(x) \in T_x N$ onto $T_x M$ for all $x \in M$ and $1 \leq i \leq k$.*

Proof. Let $\pi : TN \rightarrow N$ be the tangent bundle for N . Then $(e_i|_M)_{i=1}^k$ is a moving orthonormal basis for the vector bundle $\pi|_{\pi^{-1}(M)} : \pi^{-1}(M) \rightarrow M$, which contains TM as a sub-bundle. We may thus apply Theorem 5. \square

For example, the two dimensional sphere S^2 does not have a moving basis for its tangent space, as it does not have a nowhere zero vector field. However, if we consider \mathbb{R}^3 to be a Riemannian manifold with the Riemannian metric given by the dot product, then $(e_i)_{i=1}^3$ is a moving orthonormal basis for $T\mathbb{R}^3$, where $e_1(x, y, z) = (1, 0, 0)$, $e_2(x, y, z) = (0, 1, 0)$, and $e_3(x, y, z) = (0, 0, 1)$ for all $(x, y, z) \in \mathbb{R}^3$. We can then project $(e_i)_{i=1}^3$ onto the tangent bundle of the unit sphere to obtain a moving Parseval frame $(f_i)_{i=1}^3$ for TS^2 . In this case, $(f_i)_{i=1}^3$ will be defined by $f_1(x, y, z) = (1 - x^2, -xy, -xz)$, $f_2(x, y, z) = (-xy, 1 - y^2, -yz)$, and $f_3(x, y, z) = (-xz, -yz, 1 - z^2)$ for all $(x, y, z) \in S^2$.

If M is an n dimensional smooth manifold, and $\phi : M \rightarrow N$ is an embeds into a k dimensional Riemannian manifold N with a moving orthonormal basis for TN , then we can project the moving orthonormal basis onto $T\phi(M)$ and then pull it back to obtain a moving Parseval frame for TM of k vectors. Furthermore, this may be done if ϕ is only an immersion instead of an embedding. The Whitney immersion theorem gives that for all $n \geq 2$, every n dimensional paracompact smooth manifold immerses in \mathbb{R}^{2n-1} . Thus every n dimensional smooth manifold has a moving Parseval frame for its tangent bundle of $2n - 1$ vectors. When considering $n = 2$, we have that \mathbb{R}^2 , the cylinder and the torus are the only two dimensional manifolds with continuous moving basis for its tangent bundle. However, every two dimensional paracompact smooth manifold has a moving Parseval frame of three vectors obtained by immersing the manifold in \mathbb{R}^3 . Unfortunately, obtaining a moving Parseval frame in this way often does not lend us much intuition about the space in question. We present here an intuitive moving Parseval frame for the tangent bundles of the Möbius strip and Klein bottle which cannot be obtained by immersing in \mathbb{R}^3 with the usual orthonormal basis, but which reflects the topology of the surface.

Example 7. We represent the Möbius strip and Klein bottle in the standard way with the square $[0, 1] \times [0, 1]$, where we identify the top and bottom according to $(x, 1) \equiv (1 - x, 0)$ for all $0 \leq x \leq 1$, and for the Klein bottle we identify the sides according to $(1, y) \equiv (0, y)$ for all $0 \leq y \leq 1$, as seen in Figure 1.

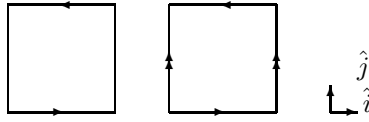


FIGURE 1. Aligned Möbius Strip (left) and Klein bottle (right)

For all $(x, y) \in [0, 1] \times [0, 1]$, let

$$f_1(x, y) = (\cos(\pi y), 0) \quad f_2(x, y) = (\sin(\pi y), 0) \quad f_3(x, y) = (0, 1).$$

It is easy to see that $(f_i)_{i=1}^3$ is a moving Parseval frame for both the Möbius strip and the Klein bottle, which naturally shows the twist in their topology.

Given a vector bundle $\pi_1 : E_1 \rightarrow M$, it is a classic problem in differential topology to find a vector bundle $\pi_2 : E_2 \rightarrow M$ so that $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$ has a moving basis. This is of course closely related to our work. Before proving Theorem 2, we need to show that our condition that $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$ has an orthonormal basis which projects to a given Parseval frame for $\pi_1 : E_1 \rightarrow M$ is

in fact stronger in general than the condition that $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$ simply has a basis. Thus the dilation theorems for moving Parseval frames do not follow as corollaries from known results in differential topology. This will be illustrated by the following simple example.

Example 8. We define a moving Parseval frame $(f_i)_{i=1}^3$ for the vector bundle $S^2 \times \mathbb{R}$ by $f_1 \equiv 1$ and $f_2 \equiv f_3 \equiv 0$. The normal bundle to $TS^2 \subset T\mathbb{R}^3$ is simply $S^2 \times \mathbb{R}$, and thus $(S^2 \times \mathbb{R}) \oplus TS^2 \cong S^2 \times \mathbb{R}^3$ has a moving basis. However, we claim that there does not exist a moving basis $(e_i)_{i=1}^3$ for $(S^2 \times \mathbb{R}) \oplus TS^2$ such that $P_{S^2 \times \mathbb{R}} e_i = f_i$ for all $i = 1, 2, 3$. Indeed, if $P_{S^2 \times \mathbb{R}} e_2 = f_2 = 0$ then $P_{TS^2} e_2$ is nowhere zero. However, S^2 does not have a nowhere zero vector field, and thus we have a contradiction.

We have a case of two vector bundles $\pi_1 : E_1 \rightarrow M$ and $\pi_2 : E_2 \rightarrow M$ and a moving Parseval frame $(f_i)_{i=1}^k$ for π_1 which does not dilate to a moving basis for $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$, even though $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$ has a moving basis of k vectors. This motivates the following question. What properties of a moving Parseval frame $(f_i)_{i=1}^k$ for a vector bundle $\pi_1 : E_1 \rightarrow M$ would guarantee that if $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$ has a moving basis of k vectors, then $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$ has a moving orthonormal basis which projects to $(f_i)_{i=1}^k$? The following theorem answers this question when $k = n + 1$, where n is the rank of the vector bundle π_1 .

Theorem 9. *Let $(f_i)_{i=1}^{n+1}$ be a moving Parseval frame for a rank n vector bundle $\pi_1 : E_1 \rightarrow M$. If $\pi_2 : E_2 \rightarrow M$ is a rank 1 vector bundle such that $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$ has a moving basis, then $\pi_2 : E_2 \rightarrow M$ has a moving Parseval frame $(g_i)_{i=1}^{n+1}$ such that $(f_i \oplus g_i)_{i=1}^{n+1}$ is a moving orthonormal basis for $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$.*

Proof. If $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$ has a moving orthonormal basis $(e_i)_{i=1}^{n+1}$, then the determinant of an operator or matrix with respect to $(e_i)_{i=1}^{n+1}$ varies smoothly over $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$. If T is an operator or matrix, we will denote $\det_e(T)$ to be the determinant of T with respect to $(e_i)_{i=1}^{n+1}$.

We will first prove the result locally, and then we will show that our local choice can actually be made globally. By Lemma 11, for each $x \in M$ there exists $\varepsilon_x > 0$ and a smoothly varying frame $(g_{x,i})_{i=1}^{n+1}$ for $\pi_2|_{\pi_2^{-1}(B_{\varepsilon_x}(x))}$ such that $(f_i \oplus g_{x,i})_{i=1}^{n+1}$ is a moving orthonormal basis for $\pi_1|_{\pi_1^{-1}(B_{\varepsilon_x}(x))} \oplus \pi_2|_{\pi_2^{-1}(B_{\varepsilon_x}(x))}$. For each $x, y \in M$, we denote $[f_i(y) \oplus g_{x,i}(y)]_{k \times k}$ to be the matrix with respect to the basis $(e_i(x))_{i=1}^{n+1}$ whose column vectors are $(f_i(y) \oplus g_{x,i}(y))_{i=1}^{n+1}$. It is easy to see that $(f_i(y) \oplus g_{x,i}(y))_{i=1}^{n+1}$ is an orthonormal basis if and only if $(f_i(y) \oplus -g_{x,i}(y))_{i=1}^{n+1}$ is an orthonormal basis. Thus without loss of generality, we may assume that $(g_{x,i})_{i=1}^{n+1}$ has been chosen such that $\det_e[f_i(x) \oplus g_{x,i}(x)]_{k \times k} = 1$ for all $x \in X$, and hence $\det_e[f_i(y) \oplus g_{x,i}(y)]_{k \times k} = 1$ for all $x \in X$ and $y \in B_{\varepsilon_x}(x)$ as \det_e is continuous. As the span of $(f_i(y) \oplus 0)_{i=1}^{n+1}$ has co-dimension 1, there is exactly one choice for $(g_{x,i}(y))_{i=1}^{n+1}$ such that $(f_i(y) \oplus g_{x,i}(y))_{i=1}^{n+1}$ is orthonormal and $\det_e[f_i(y) \oplus g_{x,i}(y)]_{k \times k} = 1$. Thus our locally smooth choice was unique, and hence is smooth globally. \square

3. PROOFS OF DILATION THEOREMS

We denote the set of all Parseval frames of k vectors for \mathbb{R}^n by $\mathcal{P}_{k,n}$. Specifically, $\mathcal{P}_{k,n} = \{(f_i)_{i=1}^k \in \oplus_{i=1}^k \mathbb{R}^n : (f_i)_{i=1}^k \text{ is a Parseval frame for } \mathbb{R}^n\}$. In order to study

moving Parseval frames over smooth manifolds, we need to first establish that $\mathcal{P}_{k,n}$ itself is a smooth manifold.

Theorem 10. *For every $k \geq n$, the set $\mathcal{P}_{k,n}$ is a smooth submanifold of $\oplus_{i=1}^k \mathbb{R}^n$ of dimension $kn - n(n+1)/2$.*

Proof. If $\mathbf{F} = (f_i)_{i=1}^k \in \oplus_{i=1}^k \mathbb{R}^n$ then the positive self-adjoint operator defined by $S_{\mathbf{F}}(x) = \sum_{i=1}^k \langle x, f_i \rangle f_i$ for all $x \in \mathbb{R}^n$ is called the frame operator for $(f_i)_{i=1}^k$. It is clear that the map $\phi : \oplus_{i=1}^k \mathbb{R}^n \rightarrow B(\mathbb{R}^n)$ given by $\phi(\mathbf{F}) = S_{\mathbf{F}}$ is smooth and $\mathcal{P}_{k,n} = \phi^{-1}(Id)$. The set of self-adjoint operators on \mathbb{R}^n is naturally diffeomorphic to $\mathbb{R}^{n(n+1)/2}$ as seen by fixing a basis and representing the self-adjoint operators by symmetric matrices. The positive definite self-adjoint operators are an open subset of the self-adjoint operators and thus form a smooth manifold. By Sard's theorem there exists a positive definite self-adjoint operator A which is a regular value of ϕ , and thus $\phi^{-1}(A)$ is a smooth submanifold of $\oplus_{i=1}^k \mathbb{R}^n$. We have that $\phi^{-1}(A)$ has dimension $kn - n(n+1)/2$ as the manifold of positive definite self adjoint operators has dimension $n(n+1)/2$ and \mathbb{R}^{kn} has dimension kn . We define a diffeomorphism $\psi_A : \oplus_{i=1}^k \mathbb{R}^n \rightarrow \oplus_{i=1}^k \mathbb{R}^n$ by $\psi_A((f_i)_{i=1}^k) = (A^{-1/2}f_i)_{i=1}^k$. As A is self adjoint we have that

$$\phi \circ \psi_A(\mathbf{F})(x) = \sum_{i=1}^k \langle x, A^{-\frac{1}{2}}f_i \rangle A^{-\frac{1}{2}}f_i = A^{-\frac{1}{2}} \sum_{i=1}^k \langle A^{-\frac{1}{2}}x, f_i \rangle f_i = A^{-\frac{1}{2}}\phi(\mathbf{F})A^{-\frac{1}{2}}x.$$

Thus $\phi \circ \psi_A(\mathbf{F}) = A^{-\frac{1}{2}}\phi(\mathbf{F})A^{-\frac{1}{2}}$, and hence $\psi_A(\phi^{-1}(A)) = \phi^{-1}(Id)$. We conclude that $\mathcal{P}_{k,n} = \phi^{-1}(Id)$ is diffeomorphic to $\phi^{-1}(A)$ and is hence a smooth submanifold of $\oplus_{i=1}^k \mathbb{R}^n$ of dimension $kn - n(n+1)/2$. \square

We note that the same proof gives that the set of frames of k -vectors in \mathbb{R}^n with a given invertible frame operator is a smooth sub-manifold of $\oplus_{i=1}^k \mathbb{R}^n$. We now prove that locally, we can smoothly choose complementary frames for Parseval frames. Note that for $k \geq 1$, $\mathcal{P}_{k,k}$ is the collection of all orthonormal bases for \mathbb{R}^k .

Lemma 11. *For every $k \geq n$ and $F \in \mathcal{P}_{k,n}$ there exists some $\varepsilon > 0$ and a smooth map $\phi : B_{\varepsilon}(F) \cap \mathcal{P}_{k,n} \rightarrow \mathcal{P}_{k,k-n}$ such that $G \oplus \phi(G) \in \mathcal{P}_{k,k}$ for all $G \in B_{\varepsilon}(F) \cap \mathcal{P}_{k,n}$.*

Proof. We choose $H \in \mathcal{P}_{k,k-n}$ such that $F \oplus H \in \mathcal{P}_{k,k}$ and fix an orthonormal basis for \mathbb{R}^n . As mentioned earlier, this is equivalent to the matrix $[F \oplus H]_{k \times k}$ being unitary. The set of invertible matrices is open, and thus there exists $\varepsilon > 0$ such that $[G \oplus H]_{k \times k}$ is invertible for all $G \in B_{\varepsilon}(F) \cap \mathcal{P}_{k,n}$. For each $G \in B_{\varepsilon}(F)$, we apply the Gram-Schmidt procedure to the rows of $[G \oplus H]_{k \times k}$, where the procedure is applied to the rows of $[G]_{n \times k}$ before the rows of $[H]_{(k-n) \times k}$. As $G \in \mathcal{P}_{k,n}$, the rows of $[G]_{n \times k}$ are orthonormal. Thus the Gram-Schmidt procedure when applied to $[G \oplus H]_{k \times k}$ will leave the rows contained in $[G]_{n \times k}$ fixed, and hence the matrix resulting from applying the Gram-Schmidt procedure will be of the form $[G \oplus \phi(G)]_{k \times k}$ for some $\phi(G) \in \mathcal{P}_{k,k-n}$. Furthermore, the map $\phi : \mathcal{P}_{k,n} \rightarrow \mathcal{P}_{k,k-n}$ is smooth as the Gram-Schmidt procedure is smooth when applied to the rows of any set of invertible matrices. \square

The frame given in Example 8 shows that Lemma 11 is not true globally for $k = 3$ and $n = 1$. However, the proof of Theorem 9 gives that Lemma 11 is true globally for $k = n + 1$. We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $\pi_1 : E_1 \rightarrow M$ be a rank n vector bundle over a paracompact manifold M with a moving Parseval frame $(f_i)_{i=1}^k$. By Lemma 11, we can locally choose a complementary moving Parseval frame. That is, for each $x \in M$, there exists $\varepsilon_x > 0$ and a moving Parseval frame $(g_{x,i})_{i=1}^k$ for the trivial vector bundle $\pi_x : B_{\varepsilon_x}(x) \times \mathbb{R}^{k-n} \rightarrow B_{\varepsilon_x}(x)$ such that $(f_i \oplus g_i)_{i=1}^k$ is a moving orthonormal basis for the vector bundle $\pi_1|_{\pi_1^{-1}(B_{\varepsilon_x}(x))} \oplus \pi_x$. The collection of sets $\{B_{\varepsilon_x}(x)\}_{x \in M}$ is an open cover of M , and thus there is a partition of unity $\{\psi_a\}_{a \in A}$ subordinate to a locally finite open refinement $\{U_a\}_{a \in A}$. We thus have for each $a \in A$ a moving Parseval frame $(g_{a,i})_{i=1}^k$ for the trivial vector bundle $\pi_a : U_a \times \mathbb{R}^{k-n} \rightarrow U_a$ such that $(f_i \oplus g_{a,i})_{i=1}^k$ is a moving orthonormal basis for $\pi_1|_{\pi_1^{-1}(U_a)} \oplus \pi_a$. We use the partition of unity to extend $(g_{a,i})_{i=1}^k$ to all of M by defining $g_i = \bigoplus_{a \in A} \psi_a^{1/2} g_{a,i}$ for all $1 \leq i \leq n$, where we set $g_{a,i}(x) = 0$ if $x \notin U_a$. Thus g_i is a smooth section of the trivial vector bundle $\pi_t : M \times \bigoplus_{a \in A} \mathbb{R}^{k-n} \rightarrow M$ for all $1 \leq i \leq k$. The following simple calculations show that $(f_i(x) \oplus g_i(x))_{i=1}^k$ is an orthonormal set of vectors in $\pi_1^{-1}(x) \oplus \bigoplus_{a \in A} \mathbb{R}^{k-n}$ for all $x \in M$. For all $1 \leq i, j \leq k$, we have the following calculation.

$$\begin{aligned} \langle f_i(x) \oplus g_i(x), f_j(x) \oplus g_j(x) \rangle &= \langle f_i(x), f_j(x) \rangle + \sum_{a \in A} \psi_a \langle g_{a,i}(x), g_{a,j}(x) \rangle \\ &= \sum_{a \in A} \psi_a (\langle f_i(x), f_j(x) \rangle + \langle g_{a,i}(x), g_{a,j}(x) \rangle) \\ &= \sum_{a \in A} \psi_a \langle f_i(x) \oplus g_{a,i}(x), f_j(x) \oplus g_{a,j}(x) \rangle \\ &= \sum_{a \in A} \psi_a \delta_{i,j} = \delta_{i,j} \end{aligned}$$

Thus, $(f_i \oplus g_i)_{i=1}^k$ is a sequence of smooth orthonormal sections of $\pi_1 \oplus \pi_t$, and hence $E := \text{span}_{1 \leq i \leq k, x \in M} f_i(x) \oplus g_i(x)$ is a smooth manifold and we have an induced vector bundle $\pi_E : E \rightarrow M$. We now show that $(f_j)_{1 \leq j \leq k}$ being a moving Parseval frame implies that $\text{span}_{1 \leq j \leq k} f_j(x) \oplus 0 \subset \text{span}_{1 \leq j \leq k} f_j(x) \oplus g_j(x)$ for all $x \in M$. Indeed, if $x \in M$ and $y \in \pi_1^{-1}(x)$ then we calculate the following.

$$\|P_{\pi_E^{-1}(x)} y \oplus 0\|^2 = \sum_{i=1}^k \langle y \oplus 0, f_i \oplus g_i \rangle^2 = \sum_{i=1}^k \langle y, f_i \rangle^2 = \|y\|^2$$

Which implies that $y \oplus 0 = P_{\pi_E^{-1}(x)} y \oplus 0$. Thus we conclude that $\text{span}_{1 \leq j \leq k} f_j(x) \oplus 0 \subset \text{span}_{1 \leq j \leq k} f_j(x) \oplus g_j(x)$, and hence $p_1 : E_1 \rightarrow M$ is a sub-bundle of $\pi_E : E \rightarrow M$. It is then possible to define $\pi_2 : E_2 \rightarrow M$ as the orthogonal bundle of $\pi_1 : E_1 \rightarrow M$ in $\pi_E : E \rightarrow M$. We have that $f_i = P_{E_1} f_i \oplus g_i$ and thus by definition $g_i = P_{E_2} f_i \oplus g_i$. As the orthogonal projection of a moving orthonormal basis onto a sub-bundle, $(g_i)_{i=1}^\infty$ is a moving Parseval frame for E_2 and is a complementary frame for $(f_i)_{i=1}^\infty$. \square

The above proof takes an approach using local coordinates, and then combines the pieces using a partition of unity. As this is a common technique in differential topology, the above construction is valuable in that it could potentially be combined with other proofs and constructions. We present as well a second proof which avoids local coordinates and is based on the original proof of the dilation theorem of Han and Larson.

Second proof of Theorem 2. Let $\pi_1 : E_1 \rightarrow M$ be a rank n vector bundle over a paracompact manifold M with a moving Parseval frame $(f_i)_{i=1}^k$. Let $\pi : M \times \mathbb{R}^k \rightarrow M$ denote the trivial rank k vector bundle over M , and let $(e_i)_{i=1}^k$ be the moving unit vector basis for $\pi : M \times \mathbb{R}^k \rightarrow M$. We define a bundle map $\theta : E_1 \rightarrow M \times \mathbb{R}^k$ over M by $\theta(y) = \sum_{i=1}^k \langle y, f_i(\pi_1(y)) \rangle e_i(\pi_1(y))$. As $(f_i(\pi_1(y)))_{i=1}^k$ is a Parseval frame for the fiber containing y , we have that $\|y\|^2 = \sum_{i=1}^k \langle y, f_i(\pi_1(y)) \rangle^2 = \|\theta(y)\|^2$. Hence, $\theta|_{\pi_1^{-1}(x)}$ is a linear isometric embedding of the fiber for $\pi_1^{-1}(x)$ into the fiber $\pi^{-1}(x)$ all $x \in M$. Thus, for convenience, we may identify the bundle $\pi_1 : E_1 \rightarrow M$ with $\pi|_{\theta(E_1)} : \theta(E_1) \rightarrow M$. In particular, we have that

$$(1) \quad \langle y, e_i(\pi(y)) \rangle = \langle y, f_i(\pi(y)) \rangle \text{ for all } y \in E_1.$$

Let $P_1 : M \times \mathbb{R}^k \rightarrow E_1$ be orthogonal projection, that is, P_1 is the bundle map such that $P_1(y)$ is the orthogonal projection of y onto the fiber $\pi_1^{-1}(\pi(y))$. We now show that $P_1 \circ e_i = f_i$ for all $1 \leq i \leq k$. We let $x \in M$, $y \in \pi_1^{-1}(x)$ and $1 \leq i \leq k$, and obtain

$$\begin{aligned} \langle y, P_1(e_i(x)) \rangle &= \langle y, e_i(x) \rangle && \text{as } y \in E_1 \\ &= \langle y, f_i(x) \rangle && \text{by the definition of } \theta \text{ as } x = \pi_1(y) \end{aligned}$$

Thus $\langle y, P_1(e_i(x)) \rangle = \langle y, f_i(x) \rangle$ for all $y \in \pi_1^{-1}(x)$, and hence $P_1(e_i(x)) = f_i(x)$ for all $x \in M$. \square

We now consider the case where M is a smooth paracompact manifold with a moving Parseval frame for its tangent bundle TM . We may apply Theorem 2 to obtain a vector bundle containing TM with a moving orthonormal basis which projects to the moving Parseval frame. However, we want the moving orthonormal basis to be for a larger tangent bundle and not just a general vector bundle. To do this, we will show that actually the total space of the vector bundle given by Theorem 2 will be a Riemannian manifold with an orthonormal basis which projects to the given Parseval frame for TM .

Proof of Theorem 3. We denote $\pi_1 : TM \rightarrow M$ to be the tangent bundle. By Theorem 2 there exists a rank $(k - n)$ vector bundle $\pi_2 : N \rightarrow M$ with a moving Parseval frame $(g_i)_{i=1}^k$ so that $(f_i \oplus g_i)_{i=1}^k$ is a moving orthonormal basis for the vector bundle $\pi_1 \oplus \pi_2$. The manifold N has dimension k , as M has dimension n and the vector bundle $\pi_2 : N \rightarrow M$ has rank $k - n$. For $p \in N$ and $\gamma : \mathbb{R} \rightarrow N$ such that $\gamma(0) = p$, we have the differential $D_\gamma \in TN$ defined by

$$D_\gamma(f) = \frac{d}{dt} f(\gamma(t))|_{t=0},$$

for each smooth real valued f defined on an open neighborhood of p . We define a smooth map $\Theta : N \times N \rightarrow N$ by

$$\Theta(p, q) = \sum_{i=1}^k \langle g_i(\pi_2(p)), p \rangle g_i(\pi_2(q)) \quad \text{for all } p, q \in N.$$

The map Θ has been constructed so that if p and q are contained in the same fiber of $\pi_2 : N \rightarrow M$, i.e. $\pi_2(q) = \pi_2(p)$, then $\Theta(p, q) = p$. Note that if $q_0, q_1 \in N$ such that $\pi_2(q_0) = \pi_2(q_1)$, then $\Theta(p, q_0) = \Theta(p, q_1)$. As $\pi_2(\Theta(p, q)) = \pi_2(q)$, we thus have that $\Theta(p, q) = \Theta(p, \Theta(p, q))$ for all $p, q \in N$. We use Θ to define a smooth map

$\psi : TN \rightarrow TN$ by setting for each smooth $\gamma : \mathbb{R} \rightarrow N$, $\psi(D_\gamma) = D_\gamma - D_{\Theta(\gamma(0), \gamma)}$. In other words, if $\gamma : \mathbb{R} \rightarrow N$ is smooth and f is a smooth real valued function defined on an open neighborhood of $\gamma(0)$, then

$$\psi(D_\gamma)(f) = \frac{d}{dt}f(\gamma(t))|_{t=0} - \frac{d}{dt}f\left(\sum_{i=1}^k \langle g_i(\pi_2(\gamma(0))), \gamma(0) \rangle g_i(\pi_2(\gamma(t)))\right)|_{t=0}.$$

Note that if $D_{\gamma_0} = D_{\gamma_1}$ then $\psi(D_{\gamma_0}) = \psi(D_{\gamma_1})$, and that $\psi(aD_{\gamma_0} + D_{\gamma_1}) = a\psi(D_{\gamma_0}) + \psi(D_{\gamma_1})$ for all $a \in \mathbb{R}$ and smooth $\gamma_0, \gamma_1 : \mathbb{R} \rightarrow N$. Furthermore, if $\pi_2(\gamma)$ is constant, then $\psi(D_\gamma) = D_\gamma$.

For each $p \in N$, we define a linear operator $\Phi : T_p N \rightarrow T_{\pi_2(p)} M$ by $\Phi(D_\gamma) = D_{\pi_2 \circ \gamma}$. Then $\Phi(\psi(\gamma)) = 0$ for all smooth $\gamma : \mathbb{R} \rightarrow N$, as

$$\Phi(\psi(\gamma)) = D_{\pi_2 \circ \gamma} - D_{\pi_2(\Theta(\gamma(0), \gamma))} = D_{\pi_2 \circ \gamma} - D_{\pi_2 \circ \gamma} = 0.$$

For each $q \in \pi_2^{-1}(\pi_2(p))$ we define D_q as the derivative at p in the direction of q . That is, $D_q(f) = \frac{d}{dt}f(p + tq)|_{t=0}$, for each smooth real valued f defined on an open neighborhood of p . We have that $\psi(D_q) = D_q$ for all $q \in \pi_2^{-1}(\pi_2(p))$ as $\pi_2(p + tq)$ is constant with respect to t . Thus $\{D_q\}_{\pi_2(q)=\pi_2(p)} = \psi(T_p N) = \Phi^{-1}(0)$ as $\{D_q\}_{\pi_2(q)=\pi_2(p)} \subseteq \psi(T_p N) \subseteq \Phi^{-1}(0)$ and both spaces $\{D_q\}_{\pi_2(q)=\pi_2(p)}$ and $\Phi^{-1}(0)$ are $(k - n)$ -dimensional. For each smooth real valued f defined on an open neighborhood of p , we denote q_γ to be the unique vector in $\pi_2^{-1}(\pi_2(p))$ such that $D_{q_\gamma} = \psi(D_\gamma)$.

We define a smooth bundle map $\phi : TN \rightarrow E(\pi_1 \oplus \pi_2)$ by for $\gamma : \mathbb{R} \rightarrow N$, we set $\phi(D_\gamma) = D_{\pi_2 \circ \gamma} \oplus q_\gamma$. Note that for each $p \in N$, $\phi|_{T_p N} : T_p N \rightarrow T_{\pi_2(p)} M$ is an isomorphism as $D_{\pi_2 \circ \gamma} = 0$ if and only if $D_\gamma = D_{q_\gamma}$. Thus ϕ induces a Riemannian metric $\langle \cdot, \cdot \rangle_N$ on TN by $\langle f, g \rangle_N = \langle \phi(f), \phi(g) \rangle$. We have that $(f_i \oplus g_i)_{i=1}^k$ is a moving orthonormal basis for $\pi_1 \oplus \pi_2$, and hence $(\phi^{-1}(f_i \oplus g_i))_{i=1}^k$ is a moving orthonormal basis for TN . If $p \in M$ and $\gamma : \mathbb{R} \rightarrow M \subset N$ such that $\gamma(0) = p$ then $\phi(D_\gamma) = D_\gamma \oplus 0$. Thus $\phi^{-1}(f_i(p) \oplus 0) = f_i(p)$ for all $1 \leq i \leq k$ and $p \in M$, and hence $f_i = P_{T_p M} \phi^{-1}(f_i \oplus g_i)$ for all $1 \leq i \leq k$. \square

We may apply Theorem 3 to obtain the following corollary, where we call a smooth manifold parallelizable if it has a moving basis for its tangent bundle.

Corollary 12. *Let M be a paracompact smooth manifold. If M immerses in a k dimensional parallelizable paracompact smooth manifold, then M embeds in a k dimensional parallelizable paracompact smooth manifold.*

Proof. Assume that M immerses in a parallelizable smooth manifold N . We may assign a Riemannian metric to N such that the moving basis for TN is an orthonormal basis. Projecting this orthonormal basis then pulling back to TM gives a moving Parseval frame for TM of k vectors. Thus M embeds in a k dimensional parallelizable smooth manifold by Theorem 3. \square

4. OPEN PROBLEMS

The manifold N constructed in Theorem 3 is the total space of a vector bundle, and is hence not compact. We thus have the following question:

Question 13. Let M^n be a smooth n -dimensional compact Riemannian manifold and $(f_i)_{i=1}^k$ be a moving Parseval frame for TM for some $k \geq n$. Does there

exists a smooth k -dimensional compact Riemannian manifold N^k which has a moving orthonormal basis $(e_i)_{i=1}^k$ such that N^k contains M^n as a submanifold and $P_{T_x M} e_i(x) = f_i(x)$ for all $x \in M^n$ and $1 \leq i \leq k$?

If, $(f_i)_{i=1}^k$ is a Parseval frame for \mathbb{R}^n and $k > m > n$, then there exists a Parseval frame $(h_i)_{i=1}^k$ for $\mathbb{R}^n \oplus \mathbb{R}^{m-n}$ such that $P_{\mathbb{R}^n}(h_i) = f_i$ for all $1 \leq i \leq m$. Thus instead of dilating all the way to an orthonormal basis for a k -dimensional Hilbert space, it is possible to dilate to a Parseval frame for a m -dimensional Hilbert space. This motivates the following question in the vector bundle setting.

Question 14. Let $k > m > n$ be integers, and let $\pi_1 : E_1 \rightarrow M$ be a rank n vector bundle over a paracompact manifold M with a moving Parseval frame $(f_i)_{i=1}^k$. Does there exist a rank $m - n$ vector bundle $\pi_2 : E_2 \rightarrow M$ with a moving Parseval frame $(g_i)_{i=1}^k$ so that $(f_i \oplus g_i)_{i=1}^k$ is a moving Parseval frame for the vector bundle $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \rightarrow M$?

In [BF], it is proven that for every natural numbers $k \geq n$, there exists a tight frame $(f_i)_{i=1}^k$ of \mathbb{R}^n such that $\|f_i\| = 1$ for all $1 \leq i \leq k$, which they call a finite unit tight frame or FUNTF. An explicit construction for FUNTFs is given in [DFKLOW], and they are further studied in [BC],[CFM]. FUNTFs are also of interest for applications in signal processing, as they minimize mean squared error under an additive noise model for quantization [GK]. For a vector bundle to have a moving FUNTF, it is necessary that it have a nowhere zero section. Thus we have the following question concerning moving FUNTFs.

Question 15. Let $\pi : E \rightarrow N$ be a rank n vector bundle over a paracompact manifold N such that π has a nowhere zero section. For what $k \geq n$ does π have a moving tight frame $(f_i)_{i=1}^k$ such that $\|f_i(x)\| = 1$ for all $x \in N$ and $1 \leq i \leq k$?

These questions are general and potentially difficult, and so solutions for certain cases would still be valuable. For instance, for general values of k and n , it is unknown if the collection of FUNTFs are connected [DySt]. A moving FUNTF of k sections for a rank n vector bundle over the circle can be thought of as a path in the collection of FUNTFs of k vectors for \mathbb{R}^n . Thus, knowing whether or not a rank n vector bundle over the circle has a FUNTF of k vectors will give insight into the problem of determining the connected components of the collection of FUNTFs of k vectors for \mathbb{R}^n .

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